MATH 4E Fall 2012
Handout for Sections 5.5, 5.6

- Theorem: Division Algorithm for Polynomials
  If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is great than zero, then there are unique polynomial fuctions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or } f(x) = q(x)g(x) + r(x)$$

  We can think of $f(x)$ as our dividend, $q(x)$ as our quotient, $g(x)$ as the divisor, and $r(x)$ the remainder.

- Remainder Theorem:
  Let f be a polynomial function. Then if $f(x)$ is divided by $x - c$ then the remainder is $f(c)$.

- Factor Theorem:
  Let f be a polynomial function. The $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$. This is if and only if! So the theorem goes both ways. That is,

$$\text{If } f(c) = 0 \text{ then } x - c \text{ is a factor of } f(x).$$

$$\text{If } x - c \text{ is a factor of } f(x) \text{ then } f(c) = 0.$$

- Theorem: Number of Real Zeros
  A polynomial function cannot have more real zeros than its degree.

- Rational Zeros Theorem:
  Let f be a polynomial function of degree or or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,\ a_0 \neq 0$$

  where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

- Steps for Finding the Real Zeros of a Polynomial Function

  1. Use the degree of the polynomial to determine the maximum number of zeros
  2. (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify possible zeros.
     (b) Use substitution, synthetic division, or long division, to test each potential zero.

     Remember to use factoring techniques you already know!

- Theorem: Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible (prime) quadratic factors.

- Theorem: A polynomial functino of odd degree that has real coefficients has at least one real zero.

- Theorem: Bounds on Zero
  Let f be a polynomial function whose leading coefficient is 1, ie

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

  A bound M on the real zeros of f is the smaller of the two numbers

$$\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\},\ 1 + \text{Max}\{|a_0|, |a_1|, \cdots, |a_{n-1}|\}$$

- Intermediate Value Theorem:
  Let f denote a polynomial function. If $a < b$ and $f(a)$ and $f(b)$ are of opposite sign, then there is at least one real zero of f between a and b.

- Definition: A variable in the complex number system is referred to as a complex variable. A complex polynomial function f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where $a_n, a_{n-1}, \cdots, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer and x is a complex variable. As before, a_n is the leading coeffecient and any complex number r such that $f(r) =$ is a complex zero of f.

- Fundamental Theorem of Algebra
  Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

- Theorem: Every complex polynomial function $f(x)$ of negree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n (x - r_1)(x - r_2) \cdots (x - r_n)$$

  where $a_n, r_1, r_2, \cdots, r_n$ are complex numbers. We can also say, every complex polynomial of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

- Conjugate Pairs Theorem:
  Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

- Corollary: A corollary is something that follows almost directly from a theorem. Because of the Conugate Pairs Theorems, we may say:
  A polynomial function f of odd degree with real coefficients has at least one real zero.

- Theorem: Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.