Let A be a real number. In order to be consistent within our system (and also because it is a very natural thing to require) we would like there to be real numbers x and y such that

$$A^i = x + iy;$$

that is, we want raising a real number to an complex number to give us a complex number. Further, suppose we can write

$$A^{-i} = x - iy.$$

Then recalling the brief mention of the magnitude (or modulus) we covered in the complex number section,

$$x^2 + y^2 = (x - iy)(x + iy) = A^i A^{-i} = 1$$

so that for *any* real number, raising it to the i-th power will give it magnitude 1. So these points must lie on the unit circle! We know from trigonometry (if you've forgotten, check out the beginning of Chapter 7 in the book for a refresher) that along the unit circle, the point (x, y) lying on the terminal side of an angle θ is equivalent to $(\cos\theta, \sin\theta)$.

Now, trust me on the next bit. Draw some diagrams too, if you think it might help! But some of the things used will either be beyond the scope of your experience (I'm sorry for that) or things you will learn soon in trig.

For very small angles, the base of the triangle formed by this angle will be very nearly 1 and using the Small Angle Approximation, we know that

$$\sin\theta \approx \theta.$$

Euler's Formula (this requires calculus or your trust in me to introduce Maclaurin Series) states

$$e^{ix} = \cos x + i \sin x.$$

So, using one of the properties from today's lecture, and assuming we have values of A close to 1 (so that $\ln A$ is small),

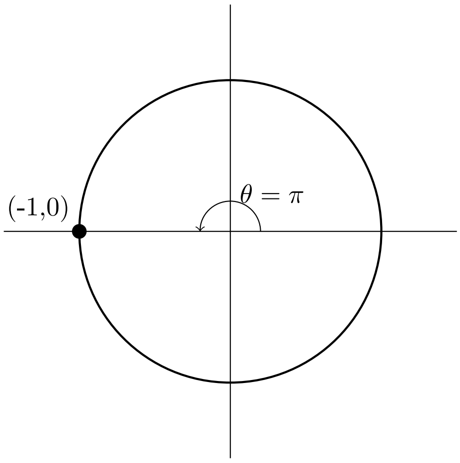
$$\begin{aligned} A^i &= e^{i \ln A} \\ &= \cos(\ln A) + i \sin(\ln A) \\ &\approx 1 + i \ln A \end{aligned}$$

So the complex number A^i can be plotted in the way we discussed earlier in a plane, or it can be considered as a point along the unit circle at an angle approximately $\ln A$.

This is all well and good, but it does not necessarily answer your question of "why is $e^{i\pi}$ negative?" And to answer that, we use Euler's Formula again!

$$e^{i\pi} = \cos\pi + i \sin\pi$$

From an earlier discussion, we know that to determine the value of these functions, we just need a unit circle and a coordinate lying on the terminal side of the angle.



Again, $\cos\theta = x$, $\sin\theta = y$. Using this information,

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 + i0 \\ &= -1 \end{aligned}$$