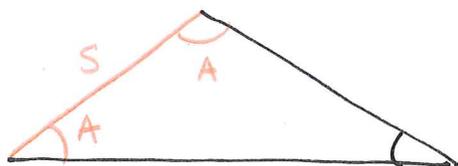


§ 9.2 - Law of Sines

There are a total of four different cases of solvable triangles:

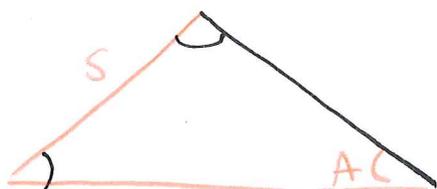


ASA

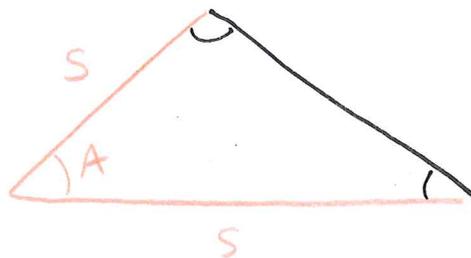


SAA

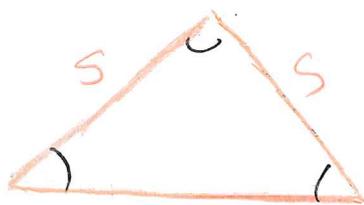
these are
← technically
"one" case,
two angles and
a side



SSA



SAS



SSS

★ Note that in these illustrations, the red indicates information we do know.

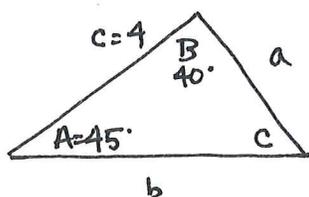
So the law of sines will be used for when we know two angles and a side (SAA or ASA) or when we know two sides and an angle opposite one of them (compare to the law of cosines which is two sides and the angle between them).

ASA Example:

$$B = 40^\circ$$

$$A = 45^\circ$$

$$c = 4$$



Since to find C we only need to do subtraction, we don't need to worry about getting two solutions. I'm not using $\sin^{-1}(x)$ to find C.

$$C = 180 - 40 - 45 = 95^\circ$$

Now to find the sides:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$a = \frac{c \sin A}{\sin C} = \frac{4 \sin 45^\circ}{\sin 95^\circ} = 2.8392$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$b = \frac{c \sin B}{\sin C} = \frac{4 \sin 40^\circ}{\sin 95^\circ} = 2.58097$$

$C =$	<u>95°</u>
$a =$	<u>2.84</u>
$b =$	<u>2.58</u>

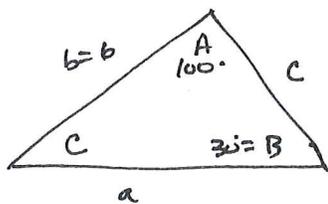
SAA Example

Note that this is essentially the same process as the previous problem, but the information we know is merely organized a little differently.

$$A = 100^\circ$$

$$B = 30^\circ$$

$$b = 6$$



$$C = \underline{50^\circ}$$

$$a = \underline{11.82}$$

$$c = \underline{9.19}$$

It's easiest to find C first

$$C = 180 - A - B = 180 - 100 - 30 = 50^\circ$$

Now to find the sides:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$a = \frac{b \sin A}{\sin B} = \frac{6 \sin(100^\circ)}{\sin(30^\circ)} = 11.81769$$

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

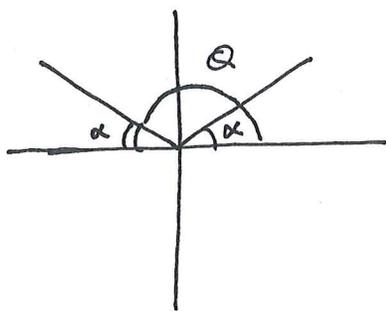
$$c = \frac{b \sin C}{\sin B} = \frac{6 \sin(50^\circ)}{\sin(30^\circ)} = 9.1925$$

when we have two sides and an angle, that's when we need to check for multiple solutions.

Anytime you're using $\sin^{-1}x$ for a physical problem, though, you should keep this in mind.

So, why two solutions?

$\sin^{-1}(x)$ outputs $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. It takes a ratio and assigns an angle to that ratio. Now, we had to restrict $\sin^{-1}x$ in this way or it would not be a proper function.



Recall from the beginning of the semester the idea of reference angles. Both α and θ will have the same sine value if the

reference angle (in this case, $180 - \theta$) is equal to α .

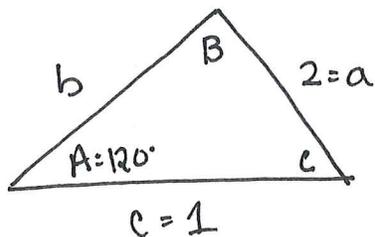
But if I were to have the sine value of θ and ask $\sin^{-1}x$ what angle it was, it would give me $\underline{\alpha}$, because θ does not live between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Since triangles are allowed to have angles which are greater than 90° (obtuse), we need to consider the option that $\sin^{-1}x$ is actually giving a reference angle rather than the angle itself.

"opposite" over "hypotenuse"

EXAMPLES

① $a=2$
 $c=1$
 $A=120^\circ$



I only have both pieces of information for A and a, so I'll be using those to find all other pieces.

* Remember: it's always safest to use information I give you as much as possible.

Since I have c, I can find C.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \frac{c \sin A}{a} = \frac{1 \sin(120)}{2} = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4}$$

$$\Rightarrow C = \sin^{-1}\left(\frac{\sqrt{3}}{4}\right) = 25.658^\circ$$

If this were a reference angle, then consider

$$C' = 180 - C = 154.341^\circ$$

But this angle is not valid! Why?

$$A + C' = 120 + 154.341 > 180$$

so that I would have nothing left to make B with.

So, the triangle has

one solution. & continue

solving with the only valid value of C.

Then,

$$B = 180 - A - C = 180 - 120 - 25.658\dots$$

$$= 34.34109^\circ$$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$b = \frac{a \sin B}{\sin A} = \frac{2 \sin(34.34109)}{\sin(120)} = 1.3027756$$

$$C = \underline{25.66^\circ}$$

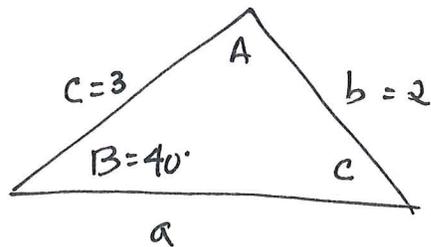
$$B = \underline{34.34^\circ}$$

$$b = \underline{1.30}$$

2. $b = 2$

$c = 3$

$B = 40^\circ$



B, b are the only pieces I have all information for so I will be using them to find all other pieces.

First, I must find C, since it is the only angle whose corresponding side I have.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\sin C = \frac{c \sin B}{b} = \frac{3 \sin(40^\circ)}{2} = 0.96418$$

$$\Rightarrow C = \sin^{-1}(0.96418) = 74.618568^\circ$$

If this were actually a reference angle then we'd have

$$C' = 180 - C = 180^\circ - 74.619^\circ = 105.3814^\circ$$

Since $B + C' < 180$, C' is a valid possibility!

Thus I have two solutions

So, first work with C, or you could say

Assume $C = 74.62^\circ$.

Then

$$A = 180^\circ - B - C = 180^\circ - 40^\circ - 74.62^\circ = 65.3814^\circ$$

So

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$a = \frac{b \sin A}{\sin B} = \frac{2 \sin(65.3814^\circ)}{\sin(40^\circ)} = 2.8286$$

Soln 1

$C = \underline{74.62^\circ}$
$A = \underline{65.38^\circ}$
$a = \underline{2.83}$

Now, suppose we're working with C' , or you could say

Assume $C = 105.38^\circ$.*

Then $A' = 180^\circ - B - C'$
 $= 180^\circ - 40^\circ - 105.38^\circ$
 $= 34.618568^\circ$

So

$$\frac{\sin A'}{a'} = \frac{\sin B}{b} \Rightarrow a = \frac{b \sin A'}{\sin B} = \frac{2 \sin(34.62^\circ)}{\sin(40^\circ)}$$

$$= 1.7676$$

Soln 2

$C' = \underline{105.38^\circ}$
$A' = \underline{34.62^\circ}$
$a' = \underline{1.77}$

*note that if you say this, no measurements should have the prime identifier.

★ I strongly encourage you to check the book examples for illustrations of the 2-triangle solutions case. It may help some of you to visualize better.

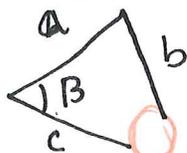
- ③ $b = 4$ Since B, b are the only pieces I have all information for, I must use them to solve the triangle.
 $c = 5$
 $B = 95^\circ$

$$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \sin C = \frac{5 \sin(95)}{4} = 1.245$$

There is no value of C which will give $\sin C > 1$
So this triangle has no solution.

Two things:

1. This means angle B is too big. b, c don't actually touch!



(I guess equiv. you could say b, c are too short)

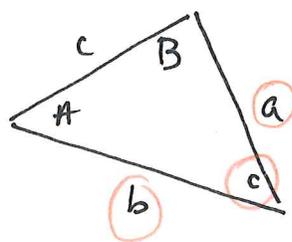
2. I expect a statement that there is no solution and why on the exam!

§ 9.3 - Law of Cosines

The Law of Cosines will cover the remaining cases for solvable triangles: SAS, SSS.

As mentioned in class, the Law of Cosines is about two sides and the angle between them:

$$c^2 = a^2 + b^2 - 2ab \cos C$$



And remember: This is the generalized Pythagorean theorem! when $C = 90^\circ$,

$$c^2 = a^2 + b^2 - 2ab \cos(90^\circ) = a^2 + b^2 \quad !$$

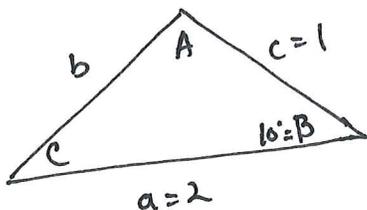
When solving a SAS triangle, that is exactly the information needed for the formula. Solving SSS means you'll just need to rearrange the formula a bit.

SAS Example

$$a = 2$$

$$c = 1$$

$$B = 10^\circ$$



I have B but not the side opposite it so I can use law of cosines:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

two sides (pointing to a and c)
angle between these sides (pointing to B)

$$= 2^2 + 1^2 - 2(2)(1) \cos(10^\circ)$$

$$= 5 - 4 \cos 10^\circ$$

$$b^2 = 1.060769$$

$$b = 1.02994$$

$$A = \underline{160.29^\circ}$$

$$C = \underline{9.71^\circ}$$

$$b = \underline{1.03}$$

At this point, you can use the law of sines to find the remaining information since I know $\frac{\sin B}{b}$. I, however, prefer the law of cosines so I will solve the triangle using it:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

looking for this (pointing to A)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(1.03)^2 + 1^2 - 2^2}{2(1.03)(1)} = -0.9414 \Rightarrow A = \cos^{-1}(-0.9414) = 160.29^\circ$$

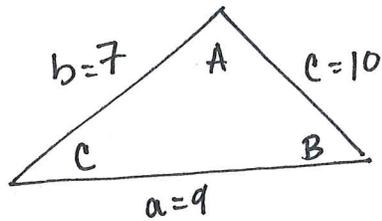
$$\Rightarrow C = 180^\circ - A - B = 180^\circ - 160.29^\circ - 10^\circ = 9.71^\circ$$

SSS Example

$$a = 9$$

$$b = 7$$

$$c = 10$$



This one I very clearly need to use law of cosines because that's the only equation I have which uses all sides at once. Where you start with this one is arbitrary.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 10^2 - 9^2}{2(7)(10)} = 0.4857$$

$$A = \cos^{-1}(0.4857) = 60.9407^\circ$$

Then,

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{9^2 + 10^2 - 7^2}{2(9)(10)} = 0.7\bar{3}$$

$$B = \cos^{-1}(0.7\bar{3}) = 42.8334^\circ$$

$$\text{So that } C = 180 - A - B = 180 - 60.9407 - 42.8334 = 76.22587$$

$$A = \underline{60.94^\circ}$$

$$B = \underline{42.83^\circ}$$

$$C = \underline{76.23^\circ}$$

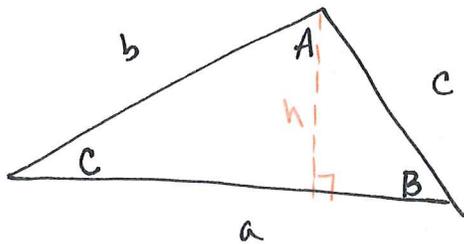
Any word problem that could be given will really be a test of your ability to visualize the problem.

Once you have the correct picture drawn, the problem is exactly like the examples done here.

I highly recommend avoiding rounding until the very end. Make use of the STORE feature on your calculator! Further, your worked out answer should have more decimal places than the answer you write in the provided blank in case there is a rounding issue.

§ 9.4 - Area of a Triangle

We all remember the formula $\frac{1}{2}bh$. And that's a great formula, but it does require finding the base or, more commonly, the height. Notice



How can I write h in terms of things I do know? I have two right triangles here! So either

$$\sin C = \frac{h}{a} \Rightarrow h = a \sin C$$

$$\text{or } \sin B = \frac{h}{c} \Rightarrow h = c \sin B$$

So that

$$K = \frac{1}{2} (\text{base}) (\text{height}) = \frac{1}{2} ac \sin B$$

So just like law of cosines, I need two sides and the angle between them.

Far more useful though is Heron's formula.

Realistically, if I were to hand you a triangle, it would be easy to measure, or at least approximate, the lengths of the sides, less so with any angle in it.

This is the power of Heron's formula: it only requires information that would be easily available.

It says

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is called the semiperimeter.

EXAMPLES

① Find the area of the triangle when $a=6$, $b=4$, $C=60^\circ$.

$$K = \frac{1}{2} ab \sin C$$

two sides (pointing to a and b) and *angle between them* (pointing to C)

$$= \frac{1}{2} (6)(4) \sin(60^\circ)^*$$

$$= 3(4) \left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{12\sqrt{3}}{2}$$

$$= 6\sqrt{3} \approx \boxed{10.39 \text{ u}^2}$$

*you can do this calculation without your calculator! Believe in yourself.

② Find the area of the triangle with sides 5, 8, and 9.

* I only have sides! I get to use Heron's formula!

$$s = \frac{1}{2}(5+8+9) = 11$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{11(11-5)(11-8)(11-9)}$$

$$= \sqrt{11(6)(3)(2)}$$

$$= \sqrt{396}$$

$$\approx \boxed{19.8997 \text{ u}^2}$$