Math 3304 Spring 2015 Exam 1

Your printed name: Solutions

Your instructor's name: ____________________

Your section (or Class Meeting Days and Time): ____________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 1 consists of this cover page, and 4 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	16	18	18	24	24	100

1. [16] Classify each differential equation by completing the columns in the following table. For each nonlinear differential equation, circle the term(s) that make it nonlinear.

Differential Equation	Order?	Linear? (Y/N)
$x^{(5)} = 2y^3 x' - 6x$	5	Y
$v'' = \sqrt{u v^2 - 1} - (1+u)\, v$	2	N
$y\,(p')^2 + (\ln y)\, p = 0$	1	N
$(r \cos r)\, w' - w = 3$	1	Y

2. a) [9] Find the general solution of $y'' - 4y' + 4y = 0$.

$$y = e^{rt} \Rightarrow r^2 - 4r + 4 = 0$$
$$\Rightarrow (r-2)^2 = 0$$
$$\Rightarrow r = 2 \text{ repeated}$$
$$\Rightarrow y = C_1 e^{2t} + C_2 t e^{2t}$$

b) [9] Find the general solution of $y'' + 4y' + y = 0$.

$$y = e^{rt} \Rightarrow r^2 + 4r + 1 = 0$$
$$\Rightarrow r = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$
$$\Rightarrow y = C_1 e^{(-2-\sqrt{3})t} + C_2 e^{(-2+\sqrt{3})t}$$

3. [18] Determine the longest interval in which the given initial value problem is certain to have a unique solution.

$$(x-3)\,y'' + (\ln x)\,y' - (\tan x)\,y = \sin x, \quad y(2) = 1, \quad y'(2) = 3.$$

Longest interval = $(\pi/2, 3)$

standard form: $y'' + \underbrace{\frac{\ln x}{x-3}}_{x \le 0,\ x=3} y' - \underbrace{\frac{\tan x}{x-3}}_{x=3} y = \underbrace{\frac{\sin x}{x-3}}_{x=3}$

functions are continuous except at

values of x where cos x = 0

$\to x = \pm\pi/2, \pm 3\pi/2, \ldots$

possible solution intervals $(0, \pi/2), (\pi/2, 3), (3, 3\pi/2), \ldots$

initial condition at $x = 2$

$x=2$ in here

$\Rightarrow (\pi/2, 3)$ is the longest interval

4. [24] Find the general solution of

$$y' + 2\frac{t+1}{t}y = 4e^{-t}y^{1/2}, \quad t > 0.$$

You may use the following fact (you do not need to prove it): For the equation $y' + p(t)\,y = q(t)\,y^n$, with $n \neq 0, 1$, the change of variable $v = y^{1-n}$ yields the equation

$$\frac{1}{1-n}v' + p(t)\,v = q(t).$$

We have $y' + p(t)y = q(t)y^n$, where

$p(t) = 2\frac{t+1}{t}$, $q(t) = 4e^{-t}$, $n = \frac{1}{2}$.

The equation for $v = y^{1-n} = y^{\frac{1}{2}}$ is

$$\underbrace{\frac{1}{1-\frac{1}{2}}}_{=2} v' + 2\frac{t+1}{t} v = 4e^{-t}$$

Divide by 2 to place in standard form for integrating factor method.

$$v' + \frac{t+1}{t}v = 2e^{-t}$$

$$IF = e^{\int \frac{t+1}{t} dt} = e^{\int 1+\frac{1}{t} dt}$$

$$= e^{t+\ln t} \quad \text{(since } t>0\text{)}$$

$$= e^t e^{\ln t} = te^t$$

$$\Rightarrow te^t v' + (t+1)e^t v = 2t$$

$$\Rightarrow (te^t v)' = 2t$$

$$\Rightarrow te^t v = \int 2t\,dt = t^2 + C$$

$$\Rightarrow v = \frac{C}{t}e^{-t} + te^{-t}$$

Now solve for y using $v = y^{\frac{1}{2}}$

$$\Rightarrow y^{\frac{1}{2}} = v = \frac{C}{t}e^{-t} + te^{-t} \Rightarrow y = \left(\frac{C}{t}e^{-t} + te^{-t}\right)^2$$

5. [24] Find the explicit solution of the initial value problem

$$y' = \frac{1}{y-2} + y - 2, \quad y(0) = -1.$$

The differential equation is nonlinear – use separation of variables. Rewrite to separate:

$$\frac{dy}{dt} = \frac{1}{y-2} + y - 2 = \frac{1}{y-2} + \frac{(y-2)^2}{y-2} = \frac{1+(y-2)^2}{y-2}$$

$$\Rightarrow \int \frac{y-2}{1+(y-2)^2}\,dy = \int dt = t + C$$

$u = 1 + (y-2)^2$

$du = 2(y-2)\,dy \to \frac{1}{2}du = (y-2)\,dy$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u}\,du = t + C \Rightarrow \frac{1}{2}\ln|u| = t + C$$

$$\Rightarrow \frac{1}{2}\ln|1+(y-2)^2| = t + C \Rightarrow \ln|1+(y-2)^2| = 2t + C$$

$$\Rightarrow |1+(y-2)^2| = e^{2t+C} = e^C e^{2t}$$

$$\Rightarrow 1 + (y-2)^2 = \pm e^C e^{2t} = C_0 e^{2t}, \quad C_0 = \text{arbitrary nonzero constant}$$

Use $y(0) = -1 \Rightarrow 1 + (-1-2)^2 = C_0 e^{2(0)} = C_0$

$$\Rightarrow C_0 = 10$$

$$\Rightarrow 1 + (y-2)^2 = 10e^{2t} \Rightarrow (y-2)^2 = 10e^{2t} - 1$$

$$\Rightarrow y - 2 = \pm\sqrt{10e^{2t}-1}$$

$$\Rightarrow y = 2 - \sqrt{10e^{2t}-1}$$

We must take the negative root in order to satisfy $y(0) = -1$