Math 3304 Spring 2015 Exam 2

Your printed name: Solutions

Your instructor's name: ______

Your section (or Class Meeting Days and Time): ______

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 2 consists of this cover page, and 4 pages of problems containing 4 numbered problems.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [22] at the beginning of a problem indicates the point value of that problem is 22. The maximum possible score on this exam is 100.

Problem	1	2	3	4	Sum
Points Earned					
Max. Points	22	26	26	26	100

1. [22] A spring hangs vertically from a rigid support. When an 8 pound box is attached to the end of the spring, the box stretches the spring 4 inches and then comes to rest. Suppose the box is given a downward displacement of 6 inches from the rest position and then is released with no initial velocity.

a) Assuming that there is no damping and that the box is acted on by an external force of $6\cos(4t)$ pounds, formulate BUT DO NOT SOLVE an initial value problem modeling the motion of the box.

$$mx'' + \gamma x' + kx = F_{ext}$$

$$x(0) = x_0, \quad x'(0) = v_0$$

usigg $g=32$ is ok!

$\underline{\gamma = 0}$ no damping

$W = mg \Rightarrow 8 = m(32.2) \Rightarrow \underline{m = \frac{8}{32.2} \quad \left(\text{or } \frac{8}{32} = \frac{1}{4}\right)}$

$W = k * \text{stretch} \Rightarrow 8 = k\left(\frac{4}{12}\right) = \frac{k}{3} \Rightarrow \underline{k = 24}$

in feet

$\underline{F_{ext}(t) = 6\cos(4t)}$

$\underline{x_0 = 1/2}$ (in feet)

$\underline{v_0 = 0}$ (no init. vel.)

b) If the external force is replaced by a force of $12\sin(\omega t)$ pounds, find the value of the constant $\omega > 0$ for which resonance occurs.

$mx'' + kx = 0 \Rightarrow \frac{1}{4}x'' + 24x = 0$ (using $g=32$ for ease)

$\Rightarrow x'' + 96x = 0$

$x = e^{rt} \Rightarrow r^2 = -96 \Rightarrow r = \pm\sqrt{96}\, i$

$\Rightarrow \underline{\omega = \sqrt{96}}$

using $g=32.2$ is ok

$\underline{\text{or}}$ $\omega = \sqrt{\frac{32.2}{8}24} = \sqrt{3 \times 32.2} = \sqrt{96.6}$

2. a) [13] Find the general solution of $y^{(5)} - 6y''' + 9y' = 0$.

$$y = e^{rt} \Rightarrow r^5 - 6r^3 + 9r = 0$$
$$\Rightarrow r(r^4 - 6r^2 + 9) = 0$$
$$\Rightarrow r(r^2-3)^2 = 0$$
$$\Rightarrow r = 0, \quad \underbrace{r^2 = 3}_{\text{repeated}} \Rightarrow \underbrace{r = \pm\sqrt{3}}_{\text{repeated}}$$

$$\Rightarrow y = C_1 + C_2 e^{\sqrt{3}t} + C_3 e^{-\sqrt{3}t} + C_4 t e^{\sqrt{3}t} + C_5 t e^{-\sqrt{3}t}$$

b) [13] Find α so that the solution of the initial value problem

$$x^2 y'' - 2y = 0, \quad y(1) = 1, \quad y'(1) = \alpha$$

is bounded as $x \to \infty$.

$$y = x^m \Rightarrow m(m-1) - 2 = 0$$
$$\Rightarrow m^2 - m - 2 = 0$$
$$\Rightarrow (m-2)(m+1) = 0$$
$$\Rightarrow m = 2, -1$$
$$\Rightarrow y = C_1 x^2 + C_2 x^{-1} \Rightarrow y' = 2C_1 x - C_2 x^{-2}$$
$$1 = y(1) = C_1 + C_2 \Rightarrow C_1 = 1 - C_2$$
$$\alpha = y'(1) = 2C_1 - C_2$$
$$\Rightarrow \alpha = 2(0) - (1)$$
$$\Rightarrow \underline{\alpha = -1}$$

need $C_1 = 0$ for $y(x)$ to be bounded as $x \to \infty$

$C_1 = 0 \Rightarrow C_2 = 1$

3. [26] Find the general solution of

$$y'' + y = \frac{1}{\sin(t)}, \quad 0 < t < \pi.$$

$y_h'' + y_h = 0 \Rightarrow y_h = e^{rt} \Rightarrow r^2 + 1 = 0$

$\Rightarrow r = \pm i \Rightarrow \boxed{y_h = C_1 \cos t + C_2 \sin t}$

$y_p = u_1 y_1 + u_2 y_2$, $y_1 = \cos t$, $y_2 = \sin t$

$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$u_1' = \dfrac{-(\sin t)\left(\frac{1}{\sin t}\right)}{(1)} \Rightarrow u_1 = -\int 1\, dt = -t$

$u_2' = \dfrac{(\cos t)\left(\frac{1}{\sin t}\right)}{(1)} \Rightarrow u_2 = \int \dfrac{\cos t}{\sin t}\, dt$

$u = \sin t$, $du = \cos t\, dt$

$\Rightarrow u_2 = \int \frac{1}{u}\, du = \ln|u| = \ln|\sin t|$

$\Rightarrow \boxed{y_p = (-t)(\cos t) + (\ln|\sin t|)(\sin t)}$

Gen soln: $\underline{y = y_h + y_p}$, both given above

4. [26] Find a particular solution of

$$y''' - y'' + y' - y = 6e^t.$$

$y_h''' - y_h'' + y_h' - y_h = 0$

$\hookrightarrow y_h = e^{rt} \Rightarrow r^3 - r^2 + r - 1 = 0$

$r = 1$ is a root $\Rightarrow$ e^t is a soln of the homogeneous problem

$\Rightarrow y_p = Ae^t$ will not work

$\Rightarrow y_p = Ate^t$ sub into the DE

$y_p' = A(te^t + e^t) = A(t+1)e^t$

$y_p'' = A[(t+1)e^t + e^t] = A(t+2)e^t$

$y_p''' = A[(t+2)e^t + e^t] = A(t+3)e^t$

The DE gives

$$\underbrace{A(t+3)e^t}_{y'''} - \underbrace{A(t+2)e^t}_{y''} + \underbrace{A(t+1)e^t}_{y'} - \underbrace{Ate^t}_{y} = 6e^t$$

$$\Rightarrow (\underbrace{3A - 2A + A}_{=2A})e^t + (\underbrace{A - A + A - A}_{=0})te^t = 6e^t$$

$\Rightarrow 2Ae^t = 6e^t \Rightarrow 2A = 6 \Rightarrow A = 3$

$\Rightarrow \boxed{y_p = 3te^t}$