Math 3304 Spring 2015 Final Exam

Your printed name: Solutions

Your instructor's name: ______________________

Your section (or Class Meeting Days and Time): ______________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Final exam consists of this cover page, 8 pages of problems containing 8 numbered problems, and 1 page of Laplace transform table.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. **Express all solutions in real-valued, simplified form.**
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [25] at the beginning of a problem indicates the point value of that problem is 25. The maximum possible score on this exam is 220.

Problem	1	2	3	4	5	6	7	8	Sum
Points Earned									
Max. Points	25	25	25	25	35	25	35	25	220

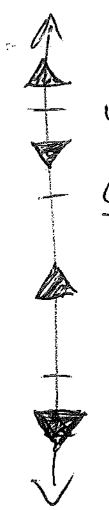
1. [25] For the differential equation $y' = (y-2)(y-3)(y+6)$,

a) Determine the equilibrium solutions (critical points) of the differential equation.
b) Sketch the phase line (or phase portrait). Be sure to **show your work.**
c) Classify each equilibrium point as either asymptotically stable, unstable, or semi-stable.
d) If $y(t)$ denotes the solution of the differential equation satisfying the initial condition $y(0) = 0$, determine $\lim_{t\to\infty} y(t)$.

(a) $0 = (y-2)(y-3)(y+6)$

$\Rightarrow \underline{y = 2, 3, -6}$

(b) (c)



3 — unstable

2 — as. stable

-6 — unstable

work for (b)

y	-7	0	$\frac{5}{2}$	4
y' = slope	$-$	$+$	$-$	$+$

$y' =$	$(y-2)$	$(y-3)$	$(y+6)$	
$\ominus$	$-$	$-$	$-$	$y=-7$
$\oplus$	$-$	$-$	$+$	$y=0$
$\ominus$	$+$	$-$	$+$	$y=\frac{5}{2}$
$\oplus$	$+$	$+$	$+$	$y=4$

(d) If $y(0)=0$, by (b) the solution $y(t)$ is increasing and must approach 2

$$\Rightarrow \lim_{t\to\infty} y(t) = 2.$$

2. Find the general solution of the following differential equations

a) [13] $y^{(5)} - 8y'' = 0$

$$y = e^{rt} \rightarrow r^5 \cancel{e^{rt}} - 8r^2 \cancel{e^{rt}} = 0$$

$$\rightarrow r^2(r^3 - 8) = 0$$

$$\rightarrow r^2(r-2)(r^2+2r+4) = 0$$

$$\rightarrow r = 0 \text{ repeated}, \quad r = 2$$

$$r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}\, i$$

$$\Rightarrow y = C_1 + C_2 t + C_3 e^{2t} + C_4 e^{-t}\cos\sqrt{3}t + C_5 e^{-t}\sin\sqrt{3}t$$

check work:

$(r-2)(r^2+2r+4)$

$= r^3 + \cancel{2r^2} + \cancel{4r}$

$\quad - \cancel{2r^2} - \cancel{4r} - 8$

$= r^3 - 8$ ✓

$r=2$ is a solution of $r^3-8=0 \rightarrow (r-2)$ is a factor

divide:

$$\begin{array}{r|l} & r^2 + 2r + 4 \\ \hline r-2 & r^3 - 8 \\ & -(r^3 - 2r^2) \\ \hline & 2r^2 \\ & -(2r^2 - 4r) \\ \hline & 4r - 8 \\ & -(4r - 8) \\ \hline & 0 \end{array}$$

b) [12] $x^2y'' + 5xy' + 4y = 0$, $x > 0$

$y = x^m \rightarrow$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$$m(m-1)\cancel{x^m} + 5m\cancel{x^m} + 4\cancel{x^m} = 0$$

$$\rightarrow m^2 - m + 5m + 4 = 0$$

$$\rightarrow m^2 + 4m + 4 = 0$$

$$\rightarrow (m+2)^2 = 0 \rightarrow m = -2 \text{ repeated}$$

$$\rightarrow y = C_1 x^{-2} + C_2 x^{-2}\ln x$$

3. [25] Solve the initial value problem

$$y' = t^2(y-2), \qquad y(0) = 4.$$

Separate variables: $\frac{dy}{dt} = t^2(y-2) \longrightarrow \frac{dy}{y-2} = t^2 dt$

$$\longrightarrow \int \frac{dy}{y-2} = \int t^2 \, dt \longrightarrow \ln|y-2| = \frac{1}{3}t^3 + C$$

$$\longrightarrow |y-2| = e^{\frac{1}{3}t^3 + C} = e^{\frac{1}{3}t^3} e^{C}$$

$$\longrightarrow y - 2 = \underbrace{\left(\pm e^{C}\right)}_{\substack{\text{arbitrary} \\ \text{nonzero} \\ \text{constant} \\ = C_0}} e^{\frac{1}{3}t^3} = C_0 e^{\frac{1}{3}t^3}$$

$$\longrightarrow y = 2 + C_0 e^{\frac{1}{3}t^3}$$

$$4 = y(0) = 2 + C_0 e^{\frac{1}{3}(0)} = 2 + C_0 \longrightarrow C_0 = 2$$

$$\longrightarrow \underline{y = 2 + 2e^{\frac{1}{3}t^3}}$$

4. [25] Find the general solution of differential equation

$$y'' + y' - 6y = 5\cos(t).$$

$y_h'' + y_h' - 6y_h = 0 \quad y_h = e^{rt} \longrightarrow r^2 + r - 6 = 0$

$\longrightarrow (r+3)(r-2) = 0 \longrightarrow r = -3, r = 2$

$\longrightarrow \underline{y_h = C_1 e^{-3t} + C_2 e^{2t}}$

Next: (MUC) $y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t, \quad y_p'' = -A\cos t - B\sin t$

sub into DE

$\longrightarrow [-A\cos t - B\sin t] + [-A\sin t + B\cos t]$

$-6[A\cos t + B\sin t] = 5\cos t$

$\longrightarrow (-A + B - 6A)\cos t + (-B - A - 6B)\sin t = 5\cos t$

match coefficients: $-A + B - 6A = 5$

$-B - A - 6B = 0 \longrightarrow A = -7B$

$\longrightarrow -7A + B = 5 \xrightarrow{A=-7B} 49B + B = 5 \longrightarrow 50B = 5 \longrightarrow \underline{B = \frac{1}{50}}$

$\longrightarrow A = -7B = \frac{-7}{50} \longrightarrow \underline{y_p = \frac{-7}{50}\cos t + \frac{1}{50}\sin t}$

Gen. soln: $\underline{y = y_h + y_p}$, y_h, y_p above

For b) ✱✱ $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$

$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$

5. a) [5] Express the function

$$g(t) = \begin{cases} t+1, & 0 \le t < 2, \\ t-1, & 2 \le t < \infty, \end{cases}$$

in terms of the unit step (Heaviside) function.

b) [30] Solve the initial value problem

$$y'' + y = 1 + u_2(t) + (t-2)u_2(t), \quad y(0) = 0, \quad y'(0) = 0.$$

a) $g(t) = (t+1)\underbrace{\begin{cases} 1, & 0 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}}_{=(1-u_2(t))} + (t-1)\underbrace{\begin{cases} 0, & 0 \le t < 2 \\ 1, & 2 \le t < \infty \end{cases}}_{=u_2(t)}$

$\longrightarrow g(t) = (t+1)(1 - u_2(t)) + (t-1)u_2(t)$

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b) Let $Y(s) = \mathcal{L}\{y\}$. $\mathcal{L}\{DE\}$ gives

$[s^2 Y(s) - \cancel{y(0)s} - \cancel{y'(0)}] + [Y(s)] = \frac{1}{s} + e^{-2s}\frac{1}{s} + \frac{1}{s^2}e^{-2s}$ ($y(0) = 0$, $y'(0) = 0$)

For the last term, used $\mathcal{L}\{f(t-2)u_2(t)\} = e^{-2s}F(s)$, where $f(t) = t$ $(\to f(t-2) = t-2)$ $\to F(s) = \frac{1}{s^2}$,

$\longrightarrow Y(s) = F(s) + e^{-2s}F(s) + e^{-2s}G(s)$, where

$F(s) = \frac{1}{s(s^2+1)}$ $\quad G(s) = \frac{1}{s^2(s^2+1)}$

PFD: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \longrightarrow 1 = A(s^2+1) + (Bs+C)s = (A+B)s^2 + Cs + A$

$\longrightarrow A = 1$, $C = 0$, $A + B = 0 \longrightarrow B = -1$

$\frac{1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} \longrightarrow 1 = (As+B)(s^2+1) + (Cs+D)(s^2) = (A+C)s^3 + (B+D)s^2 + As + B$

$\longrightarrow A+C = 0$, $B+D = 0$, $A = 0$, $B = 1 \longrightarrow D = -1$, $C = 0$

$\longrightarrow y(t) = f(t) + f(t-2)u_2(t) + g(t-2)u_2(t)$, $\quad f(t) = \mathcal{L}^{-1}\{F\}$, $g(t) = \mathcal{L}^{-1}\{G\}$ see ✱✱ above

$= [1 - \cos t] + [1 - \cos(t-2)]u_2(t) + [(t-2) - \sin(t-2)]u_2(t)$

6. [25] Solve the initial value problem

$$x' = x - 5y, \qquad x(0) = 1,$$
$$y' = x - 3y, \qquad y(0) = 1.$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, Then $\vec{x}' = \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}}_{A} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

evalues: $\det(A - \lambda I) = 0 \rightarrow 0 = \det\begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix}$

$\rightarrow 0 = (1-\lambda)(-3-\lambda) - (1)(-5) = \lambda^2 + 2\lambda - 3 + 5 = \lambda^2 + 2\lambda + 2$

$\rightarrow \lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

evectors: $\lambda = -1 + i \rightarrow (A - (-1+i)I)\vec{v} = \vec{0}$

$\rightarrow \left[\begin{array}{cc|c} 1-(-1+i) & -5 & 0 \\ 1 & -3-(-1+i) & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$

$\xrightarrow{R_2 \to (2-i)R_2 - R_1} \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad$ row 1: $(2-i)a - 5b = 0$

$\rightarrow b = \dfrac{2-i}{5} a$

$(2-i)(-2-i) - (-5)$
$= -4 \cancel{-2i} \cancel{+2i} + i^2 + 5$
$= -5 + 5 = 0$

$\rightarrow \vec{v} = \begin{bmatrix} a \\ \frac{2-i}{5} a \end{bmatrix} \underset{\text{take } a=5}{=} \begin{bmatrix} 5 \\ (2-i) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i$

gen soln: $\vec{x} = C_1 \vec{x}^{(1)} + C_2 \vec{x}^{(2)}$, $\quad \vec{x}^{(1)} = e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)$

$\vec{x}^{(2)} = e^{-t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 5 \\ 2 \end{bmatrix}\sin t\right)$

initial data: (sub in t=0)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rightarrow$ $5C_1 = 1 \rightarrow C_1 = \frac{1}{5}$

$2C_1 - C_2 = 1 \rightarrow C_2 = 2C_1 - 1 = \frac{-3}{5}$

$\rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) - \frac{3}{5} e^{-t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 5 \\ 2 \end{bmatrix}\sin t\right)$

7. [35] Find the general solution of the differential equation system

$$t\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}, \qquad t > 0.$$

You may use the following fact about the differential equation system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$, $t > 0$: If $\mathbf{x} = \mathbf{v}\,t^r$ is a solution for a constant vector $\mathbf{v}$ and a constant r, then $\mathbf{v}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\mathbf{v} = \mathbf{0}$.

evalues: $0 = \det(A - rI) = \det\begin{pmatrix} 3-r & 1 \\ 1 & 3-r \end{pmatrix} = (3-r)^2 - 1$

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad \rightarrow (3-r)^2 = 1 \rightarrow (3-r) = \pm 1 \rightarrow r = 3 \pm 1$

$\rightarrow r = 2, 4$

evectors: $(r_1 = 2)$ $\quad (A - 2I)\vec{v}^{(1)} = \vec{0} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array}\right)$ redundant eqn

$\vec{v}^{(1)} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad$ row 1: $a + b = 0 \rightarrow a = -b$

$\rightarrow \vec{v}^{(1)} = \begin{pmatrix} -b \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for $b = 1$

$(r_2 = 4)$ $\quad (A - 4I)\vec{v}^{(2)} = \vec{0} \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right)$ redundant eqn

$\vec{v}^{(2)} = \begin{pmatrix} c \\ d \end{pmatrix}$ $\quad$ row 1: $-c + d = 0 \rightarrow c = d \rightarrow \vec{v}^{(2)} = \begin{pmatrix} d \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $d = 1$

Using the given fact, two solutions are

$\vec{x}^{(1)} = \vec{v}^{(1)} t^{r_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t^2 = \begin{pmatrix} -t^2 \\ t^2 \end{pmatrix}$

$\vec{x}^{(2)} = \vec{v}^{(2)} t^{r_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 = \begin{pmatrix} t^4 \\ t^4 \end{pmatrix}$

gen soln: $\vec{x} = C_1 \vec{x}^{(1)} + C_2 \vec{x}^{(2)} = C_1 \begin{pmatrix} -t^2 \\ t^2 \end{pmatrix} + C_2 \begin{pmatrix} t^4 \\ t^4 \end{pmatrix}$

8. [25] Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}.$$

You may use that $\mathbf{x} = C_1 \begin{pmatrix} e^t \\ -3e^t \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$ is the general solution of the associated homogeneous system.

MUC: $\vec{x}_p = \vec{a} e^{-t}$ sub into DE system ($\vec{a}$ constant vector)

$$(\vec{a} e^{-t})' = A(\vec{a} e^{-t}) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t}$$

$$\rightarrow (-\vec{a})\cancel{e^{-t}} = (A\vec{a})\cancel{e^{-t}} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cancel{e^{-t}}$$

$$\rightarrow A\vec{a} + \vec{a} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(recall: $I\vec{a} = \vec{a}$)

$$\rightarrow (A + I)\vec{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

$A + I = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}$

$$\rightarrow \left[\begin{array}{cc|c} 2 & 0 & -2 \\ 3 & 3 & 1 \end{array}\right]$$

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

row 1: $2a_1 = -2 \rightarrow a_1 = -1$

row 2: $3a_1 + 3a_2 = 1 \rightarrow 3(-1) + 3a_2 = 1$

$\rightarrow 3a_2 = 4 \rightarrow a_2 = 4/3$

$$\rightarrow \vec{a} = \begin{bmatrix} 1 \\ 4/3 \end{bmatrix}$$

$$\rightarrow \vec{x}_p = \begin{bmatrix} 1 \\ 4/3 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ 4/3\, e^{-t} \end{bmatrix}$$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
7.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$f'(t)$	$sF(s) - f(0)$
9.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
10.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
11.	$u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
12.	$e^{ct}f(t)$	$F(s-c)$
13.	$\delta(t-a)$	e^{-as}
14.	$\int_0^t f(t-\tau)\, g(\tau)\, d\tau$	$F(s)\, G(s)$