Math 3304 Spring 2016 Exam 1

Your printed name: KEY

Your instructor's name: ______________________

Your section (or Class Meeting Days and Time): ______________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noise making devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 1 consists of this cover page, and 5 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] Find the general solution of

$$y' = te^{t-y}.$$

$$\frac{dy}{dt} = te^{t}e^{-y}$$

$$\int e^{y}\,dy = \int te^{t}\,dt \qquad \begin{matrix} u = t & dv = e^{t}\,dt \\ du = dt & v = e^{t} \end{matrix}$$

$$e^{y} = te^{t} - \int e^{t}\,dt$$

$$e^{y} = te^{t} - e^{t} + C$$

$$\ln e^{y} = \ln\left(te^{t} - e^{t} + C\right)$$

$$y = \ln\left(te^{t} - e^{t} + C\right)$$

2. [20] One theory of epidemic spread postulates that the time rate of change in the infected population is proportional to the product of the number of individuals who have the disease with the number of disease free individuals. Assuming that the population of mice in a certain meadow has a stable value of one thousand, use this theory of epidemic spread to write –BUT NOT SOLVE– an initial value problem that models the number $N(t)$ of infected mice at time $t \geq 0$ if ten mice were initially infected.

$$\frac{dN}{dt} = kN(1000 - N) \quad , \quad N(0) = 10$$

3. [20] A Bernoulli differential equation has the form

$$y' + p(t)y = q(t)y^n, \tag{1}$$

where n is a real number. If $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli equation (1) into the following linear differential equation

$$\frac{1}{1-n}v' + p(t)v = q(t).$$

By using this information solve the Bernoulli equation

$$ty' + y = t^2y^2, \quad t > 0$$

$$y' + \frac{1}{t}y = ty^2$$

Let $v = y^{1-2} = y^{-1}$

Then, $\frac{1}{1-2}v' + \frac{1}{t}v = t$

$$-v' + \frac{1}{t}v = t$$

$$v' - \frac{1}{t}v = -t \qquad \mu = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = t^{-1}$$

$$\frac{d}{dt}\left[t^{-1}v\right] = -1$$

$$t^{-1}v = -t + C$$

$$v = -t^2 + Ct$$

$$y^{-1} = -t^2 + Ct$$

$$y = \frac{1}{-t^2 + Ct}$$

4. [20] For the differential equation $y' = y(e^y - e^2)$,

a) Determine the equilibrium solutions (critical points) of the differential equation.
b) Sketch the phase line (or phase portrait). Be sure to **show your work.**
c) Classify each equilibrium point as either asymptotically stable, unstable, or semi-stable.
d) If $y(t)$ denotes the solution of the differential equation satisfying the initial condition $y(0) = 1$, determine $\lim_{t\to\infty} y(t)$.

a) $y' = 0$

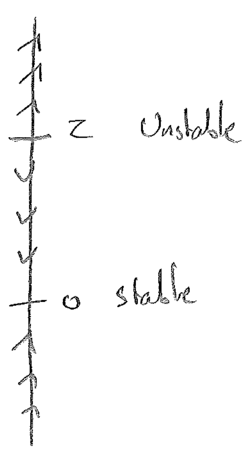
$y(e^y - e^2) = 0$

$y = 0$ or $e^y = e^2$

$y = 2$

b)

c)



$y > 2$: Try $y'(3) = 3(e^3 - e^2) > 0$

$0 < y < 2$: Try $y'(1) = 1(e^1 - e^2) < 0$

$y < 0$: Try $y'(-1) = -1(e^{-1} - e^2) > 0$

d) $\lim_{t\to\infty} y(t) = 0$

5. [20] Use the **method of undetermined coefficients** to solve the initial value problem

$$y'' - 6y' + 9y = 4e^t, \qquad y(0) = 0, \quad y'(0) = 2.$$

Homogeneous: $y'' - 6y' + 9y = 0$

let $y = e^{rt}$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$

$y = e^{3t}, te^{3t}$

$y_c = C_1 e^{3t} + C_2 te^{3t}$

Let $y_p = Ae^t$

$y_p' = Ae^t$

$y_p'' = Ae^t$

$(Ae^t) - 6(Ae^t) + 9(Ae^t) = 4e^t$

$4Ae^t = 4e^t$

$A = 1$

Thus, $y_p = e^t$

General: $y = C_1 e^{3t} + C_2 te^{3t} + e^t$

$y(0) = C_1 + 1 = 0$

$C_1 = -1$

$y = -e^{3t} + C_2 te^{3t} + e^t$

$y' = -3e^{3t} + C_2 e^{3t} + 3C_2 te^{3t} + e^t$

$y'(0) = -3 + C_2 + 0 + 1 = 2$

$-2 + C_2 = 2$

$C_2 = 4$

$$\boxed{y = -e^{3t} + 4te^{3t} + e^t}$$