Math 3304 Spring 2016 Exam 2

Your printed name: _____ KEY _____

Your instructor's name: _____

Your section (or Class Meeting Days and Time): _____

Instructions:

1. **Do not open this exam until you are instructed to begin.**

2. All cell phones and other electronic noise making devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.

3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.

4. Exam 2 consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.

5. Once the exam begins, you will have 50 minutes to complete your solutions.

6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.

7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.

8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] (Please use 32 ft/sec^2 as the acceleration of gravity in this problem) A spring, hanging vertically from a rigid support, is stretched 2 *in* by a mass that weighs 2 *lb*. The mass is pulled down an additional 3 *in* from its static equilibrium position and released with no initial velocity. Assume that no damping forces act on the mass. If the mass is acted on by a force of $3\cos 4t$, formulate, **BUT DO NOT SOLVE**, an initial value problem to describe the motion of the body.

$$L = 2in = \frac{2}{12} = \frac{1}{6} ft$$
$$mg = 2lb \implies m = \frac{mg}{g} = \frac{2}{32} = \frac{1}{16}$$
$$X(0) = 3in = \frac{3}{12} = \frac{1}{4} ft$$
$$X'(0) = 0$$
$$\beta = 0$$
$$f(t) = 3\cos 4t$$

$$k = \frac{mg}{L} = \frac{2}{\frac{1}{6}} = 12$$

$$\therefore \quad mx'' + \beta x' + kx = f$$

$$\boxed{\begin{array}{l} \frac{1}{16} x'' + 12 \cdot X = 3\cos 4t \\[2mm] X(0) = \frac{1}{4} \qquad X'(0) = 0 \end{array}}$$

2. [20] Use **the method of undetermined coefficients** to find the general solution of the differential equation

$$y^{(4)} + 4y'' = t.$$

$$y^{(4)} + 4y'' = 0$$

$$r^4 + 4r^2 = 0 \implies r^2(r^2 + 4) = 0 \implies r = 0, 0, \mp 2i$$

$$y_c = c_1 + c_2 t + c_3 \sin 2t + c_4 \cos 2t$$

$$y_p = t^2(At + B) = At^3 + Bt^2$$

$$y_p' = 3At^2 + 2Bt$$

$$y_p'' = 6At + 2B$$

$$y_p''' = 6A$$

$$y_p^{(4)} = 0$$

$$\therefore \quad 24At + 8B = t$$

$$24A = 1 \implies A = \frac{1}{24}, \quad B = 0$$

$$\therefore \quad y_p = \frac{1}{24}t^3$$

$$\therefore \quad y = y_c + y_p = c_1 + c_2 t + c_3 \sin 2t + c_4 \cos 2t + \frac{1}{24}t^3$$

3. [20] Find a particular solution of the differential equation

$$t^2 y'' - ty' + y = \frac{t}{\ln t}, \quad t > 1$$

$$t^2 y'' - ty' + y = 0$$

$$r(r-1) - r + 1 = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1, 1$$

$$y_1 = t \qquad y_2 = t \ln t$$

$$W(y_1, y_2) = \begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix} = t\ln t + t - t\ln t = t$$

$$g(t) = \frac{1}{t\ln t}$$

$$u_1 = -\int \frac{y_2 \, g}{W} \, dt = -\int \frac{t\ln t \cdot \frac{1}{t\ln t}}{t} \, dt = -\ln t$$

$$u_2 = \int \frac{y_1 \, g}{W} \, dt = \int \frac{t \cdot \frac{1}{t\ln t}}{t} \, dt = \int \frac{1}{u} \, du = \ln(\ln t)$$

$$u = \ln t$$
$$du = \frac{1}{t} dt$$

$$y_p = u_1 y_1 + u_2 y_2 = -t\ln t + t\ln t \cdot \ln(\ln t)$$

4. [20] Use **the definition of Laplace transform** to calculate the Laplace transform of the function

$$f(t) = \begin{cases} 0: & 0 \le t < 2 \\ t: & t \ge 2. \end{cases}$$

For which values of s is the Laplace transform of f defined?

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_2^\infty t e^{-st}\,dt$$

$$= \lim_{A \to \infty} \int_2^A t e^{-st}\,dt$$

$$\int_2^A t e^{-st}\,dt = -t\frac{1}{s}e^{-st}\Big|_2^A + \frac{1}{s}\int_2^A e^{-st}\,dt$$

$$\begin{aligned} u &= t \quad du = dt \\ dv &= e^{-st}\,dt \\ v &= -\frac{1}{s}e^{-st} \end{aligned}$$

$$= -A\frac{e^{-sA}}{s} + \frac{2}{s}e^{-s\cdot2} - \frac{1}{s^2}e^{-st}\Big|_2^A$$

$$= -\frac{Ae^{-As}}{s}\underset{0}{\underbrace{}} + \frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-sA}\underset{s>0}{\underbrace{}} + \frac{1}{s^2}e^{-2s}$$

$$\qquad\qquad s>0.$$

$$\therefore \mathcal{L}\{f(t)\}(s) = \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}, \qquad s>0.$$

5. [20] Find the inverse Laplace transform of $\qquad$ $F(s)$

$$e^{-3s}\frac{s-5}{s^2-4s+20}.$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\cdot F(s)\right\}\stackrel{\#12}{=\!=\!=}u_3(t)f(t-3)\qquad f(t)=?$$

$$F(s)=\frac{s-6}{s^2-4s+20}=\frac{s-6}{(s-2)^2+16}=\frac{s-2-4}{(s-2)^2+16}$$

$$=\frac{s-2}{(s-2)^2+4^2}-\frac{4}{(s-2)^2+16}$$

$$\therefore\ \mathcal{L}^{-1}\left\{F(s)\right\}=f(t)=\underset{\#9}{e^{2t}\cos 4t}-\underset{\#8}{e^{2t}\sin 4t.}$$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n, \ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2 + a^2}$
6.	$\cosh(at)$	$\dfrac{s}{s^2 - a^2}$
7.	$\sinh(at)$	$\dfrac{a}{s^2 - a^2}$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2 + b^2}$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2 + b^2}$
10.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
11.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
12.	$u_c(t) f(t-c)$	$e^{-cs} F(s)$
13.	$e^{ct} f(t)$	$F(s-c)$