Math 3304 Spring 2016 Exam 3

Your printed name: Solutions

Your instructor's name: ______________________

Your section (or Class Meeting Days and Time): ______________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noise making devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 3 consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] Given that $\mathcal{L}(t\sin at) = \dfrac{2as}{(s^2+a^2)^2}$, solve the initial value problem

$$y'' + 9y = \delta(t-3\pi) + \cos 3t, \quad y(0) = y'(0) = 0$$

by using Laplace transforms. Calculate the value of $y(\frac{7\pi}{2})$.

#10 $s^2 Y(s) - s\cancel{y(0)}^{0} + \cancel{y'(0)}^{0} + 9Y(s) = \underset{\#15}{e^{-3\pi s}} + \underset{\#5}{\dfrac{s}{s^2+9}}$

$(s^2+9)Y(s) = e^{-3\pi s} + \dfrac{s}{s^2+9}$

$Y(s) = e^{-3\pi s} \underbrace{\dfrac{1}{3}\dfrac{1 \cdot 3}{s^2+9}}_{F(s)} + \underbrace{\dfrac{1}{2\cdot 3}\dfrac{2s \cdot 3}{(s^2+9)^2}}$

$\Rightarrow f(t) = \sin 3t$ #4

$\therefore\ y(t) = u_{3\pi}(t) \cdot f(t-3\pi) + \frac{1}{6} \cdot t\sin 3t$ (which is given)

$y(t) = u_{3\pi}(t) \cdot \sin 3(t-3\pi) + \frac{1}{6} t\sin 3t.$

$y\left(\frac{7\pi}{2}\right) = 1 \cdot \sin 3\left(\frac{7\pi}{2} - 3\pi\right) + \frac{7\pi}{12} \cdot \sin 3\,\frac{7\pi}{2}$

$y\left(\frac{7\pi}{2}\right) = -1 + \frac{7\pi}{12}.$

2. [20] Find the solution of the integral equation

$$y(t) - \underbrace{\int_0^t \tau y(t-\tau)d\tau}_{t * y} = e^t.$$

$$\underset{\#3}{Y(s) - \frac{1}{s^2} \cdot Y(s)} = \underset{\#2}{\frac{1}{s-1}}$$

$$\left(1 - \frac{1}{s^2}\right) Y(s) = \frac{1}{s-1} \Rightarrow Y(s) = \frac{s^2}{(s-1)^2(s+1)}$$

$$\frac{s^2}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$$

$$s^2 = A(s-1)(s+1) + B(s+1) + C(s-1)^2$$

$$s=1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$s=0 \Rightarrow 0 = -A + B + C = -A + \frac{1}{2} + C \Rightarrow C - A = -\frac{1}{2}$$

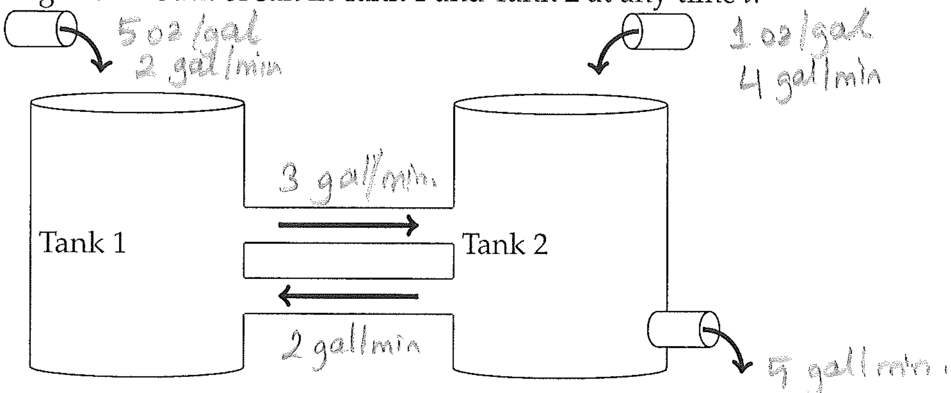
$$s=-1 \Rightarrow 1 = 4C \Rightarrow C = \frac{1}{4} \Rightarrow A = C + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$\therefore Y(s) = \frac{3/4}{s-1} + \underbrace{\frac{1/2}{(s-1)^2}}_{F(s-1)} + \frac{1/4}{s+1}$$

$$F(s-1) \Rightarrow F(s) = \frac{1/2}{s^2} \Rightarrow f(t) = \frac{1}{2} t$$

$$\therefore y(t) = \underset{\#2}{\frac{3}{4} e^t} + \underset{\#13}{e^t \frac{1}{2} t} + \underset{\#2}{\frac{1}{4} e^{-t}}$$

3. [20] Consider a system of two interconnected tanks. Tank 1 contains initially 20 gallons of pure water, and Tank 2 initially contains 50 gallons of water and 20 oz of salt. A mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 1 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 1 drains into Tank 2 at a rate 3 gallons per minute. A mixture of salt and water at a concentration of 1 ounce per gallon flows into Tank 2 at a rate of 4 gallons per minute, and the well-stirred mixture in Tank 2 drains at a rate of 7 gallons per minute, of which some flows back into Tank 1 at a rate of 2 gallons per minute. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, an initial value problem modeling the amount of salt in Tank 1 and Tank 2 at any time t.



$V_1(0) = 20$ $\qquad$ $V_2(0) = 50$

$Q_2(0) = 20$

$V_1' = 2+2-3 = 1.$ $\qquad$ $V_2' = 4+3-2-5 = 0$

$V_1 = t + C$ $\qquad$ $V_2(0) = V_2(t).$

$20 = C$

$V_1(t) = t+20$

$$Q_1' = 5\cdot 2 + \frac{Q_2}{50}\cdot 2 - \frac{Q_1}{t+20}\cdot 3, \quad Q_1(0) = 0$$

$$Q_2' = 1\cdot 4 + \frac{Q_1}{t+20}\cdot 3 - \frac{Q_2}{50}\cdot 7, \quad Q_2(0) = 20$$

$$\therefore \quad Q_1' = 10 + \frac{Q_2}{25} - 3\frac{Q_1}{t+20}, \quad Q_1(0) = 0$$

$$Q_2' = 4 + 3\frac{Q_1}{t+20} - 7\frac{Q_2}{50}, \quad Q_2(0) = 20$$

4. [20] Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}.$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \quad -3-3\lambda+\lambda+\lambda^2+8=0$$

$$\lambda^2-2\lambda+5=0$$

$$\lambda = \frac{+2 \mp \sqrt{4-4\cdot1\cdot5}}{2}$$

$$\lambda = 1 \mp 2i$$

$$\lambda = 1+2i \Rightarrow \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-2i)\xi_1 - 2\xi_2 = 0 \Rightarrow (1-i)\xi_1 = \xi_2 \Rightarrow \xi = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$X^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t(\cos 2t + i \sin 2t)$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + ie^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

$$\therefore \ X = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

5. [20] Given that $x^{(1)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ and $x^{(2)}(t) = \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}$ are two linearly independent solutions of

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x},$$

find the fundamental matrix $\Phi(t)$ such that $\Phi(0) = I$. $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$X = c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}$$

$$X^{(3)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 + 3c_2 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = 1 \\ -c_1 + 3c_2 = 0 \\ \hline -2c_2 = 1 \Rightarrow c_2 = -\frac{1}{2} \\ c_1 = \frac{3}{2} \end{array}$$

$$\therefore X^{(3)}(t) = \begin{pmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t - \frac{3}{2}e^{-t} \end{pmatrix}$$

$$X^{(4)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 + 3c_2 \end{pmatrix} \Rightarrow \begin{array}{l} -c_1 + c_2 = 0 \\ c_1 + 3c_2 = 1 \\ \hline 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \\ c_1 = -\frac{1}{2} \end{array}$$

$$\therefore X^{(4)}(t) = \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}$$

$$\therefore \Phi(t) = \begin{pmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} & -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t - \frac{3}{2}e^{-t} & -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}$$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
10.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
11.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
12.	$u_c(t)\,f(t-c)$	$e^{-cs}F(s)$
13.	$e^{ct}f(t)$	$F(s-c)$
14.	$(f * g)(t)$	$F(s)G(s)$
15.	$\delta(t-c)$	e^{-cs}