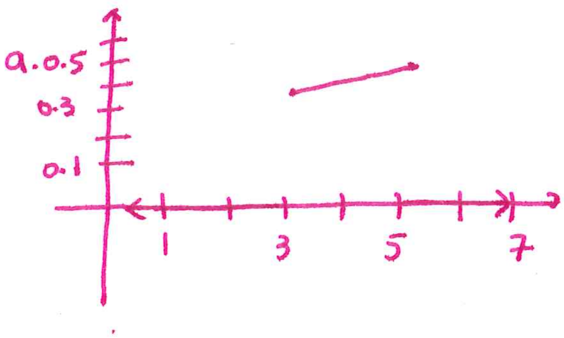
Section 4.1

1. $f(x)=\begin{cases} 0.075x+0.2, & 3 \le x \le 5 \\ 0, & \text{o.w.} \end{cases}$



$$\int_{-\infty}^{\infty} f(x)dx = \int_3^5 0.075x+0.2\,dx$$

$$= 0.0375x^2+0.2x \Big|_{x=3}^{5}$$

$$= 1.9375 - 0.9375$$

$$= 1.$$

b. $P(X \le 4) = \int_{-\infty}^{4} f(x)dx = \int_3^4 0.075x+0.2\,dx$

$$= 0.0375x^2+0.2x \Big|_{x=3}^{4}$$

$$= 1.4 - 0.9375$$

$$= \boxed{0.4625}$$

Since X is continuous, $P(X=4)=0$ and thus $P(X \le 4) = P(X<4)$.

c. $P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} 0.075x+0.2\,dx = 0.0375x^2+0.2x \Big|_{x=3.5}^{4.5}$

$$= 1.659375 - 1.159375$$

$$= \boxed{0.5}$$

$$P(X > 4.5) = \int_{4.5}^{\infty} f(x)\,dx = \int_{4.5}^{5} 0.075x + 0.2\,dx$$

$$= 0.0375x^2 + 0.2x \Big|_{x=4.5}^{5}$$

$$= 1.9375 - 1.659375$$

$$= \boxed{0.2781}$$

4. $f(x;\theta) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

a. To be a legitimate pdf, we require that $\int_{\mathbb{R}} f(x)\,dx = 1.$

Thus, $$\int_{-\infty}^{\infty} f(x;\theta)\,dx = \int_{0}^{\infty} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\,dx$$

$$= -e^{-\frac{x^2}{2\theta^2}} \Big|_{x=0}^{\infty}$$

$$= 0 + 1$$

$$= 1.$$

$\therefore f(x;\theta)$ is a valid pdf for all θ.

b. Suppose $\theta = 100$.

$$P(X \leq 200) = f(x;100) = \int_{0}^{200} \frac{x}{100^2} e^{-\frac{x^2}{2\cdot 100^2}}\,dx = -e^{-\frac{x^2}{2100^2}} \Big|_{x=0}^{200}$$

$$= -0.1353 + 1$$

$$= \boxed{0.8647}$$

$$P(\overline{X} < 200) = P(\overline{X} \leq 200) - P(\overline{X} = 200)$$
$$= \boxed{0.8647}$$

$$P(\overline{X} \geq 200) = 1 - P(\overline{X} < 200) = 1 - 0.8647 = \boxed{0.1353}$$

c. $$P(100 \leq \overline{X} \leq 200) = \int_{100}^{200} f(x; 100)\, dx$$
$$= -e^{-\frac{x^2}{2\theta^2}} \Big|_{x=100}^{x=200}$$

$$= -0.1353 + 0.6065$$
$$= \boxed{0.4712}$$

d. $$P(\overline{X} \leq x) = \int_{-\infty}^{x} f(y; \theta)\, dy = \int_{0}^{x} \frac{y}{\theta^2} e^{-\frac{y^2}{2\theta^2}}\, dy$$
$$= -e^{-\frac{y^2}{2\theta^2}} \Big|_{y=0}^{y=x}$$
$$= \boxed{1 - e^{-\frac{x^2}{2\theta^2}}, \quad x > 0}$$

6. $$f(x) = \begin{cases} k[1 - (x-3)^2], & 2 \leq x \leq 4 \\ 0, & \text{o.w.} \end{cases}$$

a.

b. We require k such that $\int_{\mathbb{R}} f(x)\,dx = 1$. Thus,

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{2}^{4} k[1-(x-3)^2]\,dx$$

$$= k\left[x - \frac{1}{3}(x-3)^3\right]_{x=2}^{4}$$

$$= k\left(\frac{11}{3} - \frac{7}{3}\right)$$

$$1 = k\,\frac{4}{3}$$

$$\therefore \boxed{k = \frac{3}{4}}$$

Now $f(x) = \begin{cases} \frac{3}{4}[1-(x-3)^2], & 2 \le x \le 4 \\ 0, & \text{o.w.} \end{cases}$

c. $P(X > 3) = \int_{3}^{\infty} f(x)\,dx = \int_{3}^{4} \frac{3}{4}[1-(x-3)^2]\,dx$

$$= \frac{3}{4}\left(x - \frac{1}{3}(x-3)^3\right)\Big|_{x=3}^{4}$$

$$= \frac{11}{4} - \frac{9}{4} = \boxed{0.5}$$ also by symmetry of the pdf!

d. $P\left(\frac{11}{4} \le X \le \frac{13}{4}\right) = \int_{\frac{11}{4}}^{\frac{13}{4}} \frac{3}{4}(1-(x-3)^2)\,dx$

$$= \frac{3}{4}\left(x - \frac{1}{3}(x-3)^3\right)\Big|_{x=\frac{11}{4}}^{\frac{13}{4}}$$

$$= 2.434 - 2.066$$

$$= \boxed{0.367}$$

e. $P(|\bar{X}-3|>0.5) = 1 - P(|\bar{X}-3|<0.5)$

$$= 1 - P(2.5 < \bar{X} < 3.5)$$

$$= 1 - \int_{2.5}^{3.5} \frac{3}{4}\left(1-(x-3)^2\right)dx$$

$$= 1 - \left[\frac{3}{4}\left(x - \frac{1}{3}(x-3)^3\right)\right]_{x=2.5}^{3.5}$$

$$= 1 - \left[2.59375 - 1.90625\right]$$

$$= \boxed{0.3125}$$

$\boxed{\text{Section 4.2}}$

12. a. $P(\bar{X}<0) = F(0) = \boxed{0.5}$

b. $P(-1<\bar{X}<1) = F(1) - F(-1) = \boxed{0.6875}$

c. $P(\bar{X}>1.5) = 1 - P(\bar{X}\le 1.5) = 1 - F(1.5) = \boxed{0.9375}$

d. $f(x) = F'(x) = \frac{d}{dx}\left(\frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\right)$

$$= 0 + \frac{3}{32}(4 - x^2)$$
$$= 0.09375(4-x^2).$$

$$\therefore f(x) = \begin{cases} 0.09375, & -2 \le x < 2 \\ 0, & \text{o.w.} \end{cases}$$

e. $\tilde{\mu}$ denotes median. By definition,

$$F(\tilde{\mu}) = 0.5.$$

From (a), we have that $F(0) = 0.5$ so that $\tilde{\mu} = 0$, as desired.

14. $f(x) = \begin{cases} \frac{1}{20-7.5}, & 7.5 \le x \le 20 \\ 0, & \text{o.w.} \end{cases}$

a. $E(X) = \int x f(x)\,dx$. Since f is a horizontal line we may find the midpoint. Thus

$$E(X) = \frac{A+B}{2} = \frac{20+7.5}{2} = \boxed{13.75}$$

$$V(X) = E(X^2) - [E(X)]^2.$$

Thus, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{7.5}^{20} 0.08x^2\,dx$

$$= 0.02\bar{6}\,x^3 \Big|_{7.5}^{20}$$

$$= 202.08\bar{3}$$

$\therefore V(X) = E(X^2) - [E(X)]^2 = 202.08\bar{3} - (13.75)^2 = \boxed{13.02}$

b. CDF $= P(\bar{X} \leq x) = F(x) = \int_{-\infty}^{x} f(y)\,dy$

$$= \int_{7.5}^{x} 0.08\,dy$$

$$= 0.08y \Big|_{7.5}^{x}$$

$$= 0.08x - 0.6$$

$$\therefore \boxed{F(x) = \begin{cases} 0 & , \quad x < 7.5 \\ 0.08x - 0.6 & , \quad 7.5 \leq x \leq 20 \\ 1 & , \quad x > 20 \end{cases}}$$

c. $P(\bar{X} \leq 10) = F(10) = 0.08(10) - 0.6 = \boxed{0.2}$

$P(10 \leq \bar{X} \leq 15) = F(15) - F(10) = 0.6 - 0.2 = \boxed{0.4}$

d. $\sigma = 3.61$ so that $\mu \pm \sigma = (10.14, 17.36)$. Thus,

$$P(10.14 \leq \bar{X} \leq 17.36) = F(17.36) - F(10.14) = 0.7888 - 0.2112$$
$$= \boxed{0.5776}$$

Similarly, $P(\mu - 2\sigma \leq \bar{X} \leq \mu + 2\sigma) = P(6.53 \leq \bar{X} \leq 20.97) = \boxed{1}$.

Reaction Time: $f(x) = \begin{cases} \frac{5}{4x^2} & , \quad 1 \leq x \leq 5 \\ 0 & , \quad \text{o.w.} \end{cases}$

a. For $x \in [1, 5]$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{1}^{x} \frac{5}{4y^2}\,dy = -\frac{5}{4y} \Big|_{y=1}^{x} = \frac{5}{4} - \frac{5}{4x}$

$$\therefore \boxed{F(x) = \begin{cases} 0 & , \quad x < 1 \\ \frac{5}{4}\left(1 - \frac{1}{x}\right) & , \quad 1 \leq x \leq 5 \\ 1 & , \quad x > 5 \end{cases}}$$

b. 1. $P(X \le 2.5) = F(2.5) = \boxed{0.75}$

2. $P(X \ge 1.5) = 1 - P(X < 1.5) = 1 - F(1.5) = \boxed{0.58\overline{3}}$

3. $P(1.5 \le X \le 2.5) = F(2.5) - F(1.5) = \boxed{0.\overline{3}}$

c. $E(X) = \int_1^5 \frac{5}{4x}\,dx = \frac{5}{4} \log x \Big|_{x=1}^{5} = \boxed{2.01}$

$E(X^2) = \int_1^5 \frac{5}{4}\,dx = \frac{5}{4} x \Big|_{x=1}^{5} = 5$

$\therefore V(X) = E(X^2) - [E(X)]^2 = 5 - (2.01)^2 = 0.9527$

$\Rightarrow \boxed{\sigma_X = 0.9760}$

d. We require $\hat{x}$ such that

$$F(\hat{x}) = 0.75.$$

Thus

$$\frac{5}{4}\left(1 - \frac{1}{\hat{x}}\right) = 0.75$$

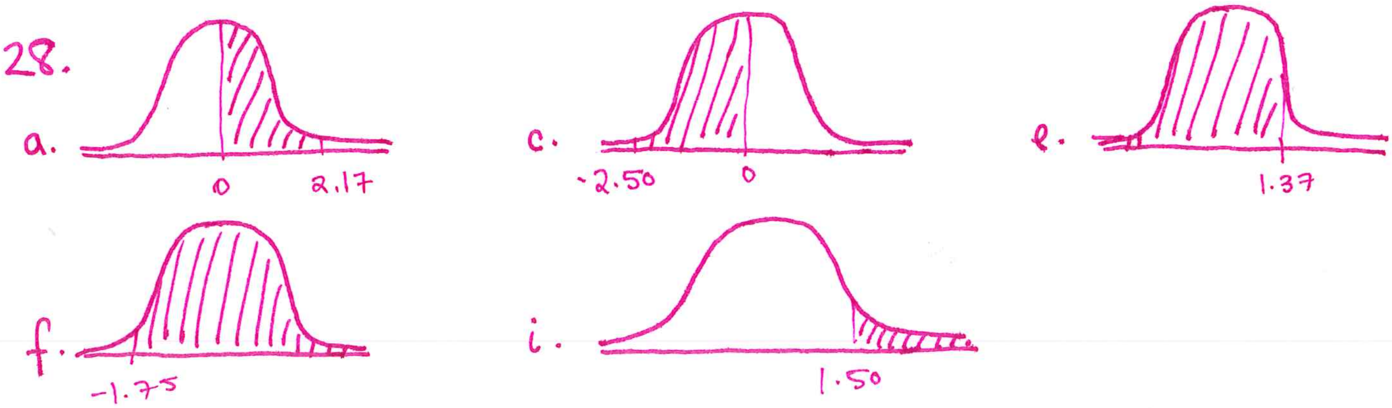
$$1 - \frac{1}{\hat{x}} = 0.4$$

$$\frac{1}{\hat{x}} = 0.6$$

$$\boxed{\hat{x} = 1.\overline{6}}$$

Section 4.3

28.



36. a. $P(X < 1500) = P(Z < 3) = \boxed{0.9987}$, from Table A3

$P(X \geq 1000) = 1 - P(X < 1000) = 1 - P(Z < -0.\bar{3}) = 1 - 0.3707 = \boxed{0.6293}$
from table A3

b. $P(1000 \leq X \leq 1500) = P(0.33 \leq Z \leq 3) = \boxed{0.628}$

c. We need the 2^{nd} percentile.

$$F(\hat{x}) = 0.02$$
$$\hat{z} = -2.05 \quad \text{"inverse" table reading}$$
$$\hat{z} = \frac{\hat{x} - \mu}{\sigma} = -2.05$$
$$\hat{x} = 742.5$$

$\boxed{\therefore \text{ only } 2\% \text{ of droplets are } 742.5\ \mu m \text{ or less!}}$

d. Let X_i be droplet i's size, $i \in \{1, 2, 3, 4, 5\}$.
Then we want

$$P(\{X_1 \geq 1500\} \cup \{X_2 \geq 1500\} \cup \cdots \cup \{X_5 \geq 1500\}) \quad \text{take complement}$$

$$= 1 - P(\{X_1 \geq 1500\} \cap \{X_2 \geq 1500\} \cap \cdots \cap \{X_5 \geq 1500\})$$

$$= 1 - P(X_1 < 1500) P(X_2 < 1500) \cdots P(X_5 < 1500) \quad \text{by independence}$$

$$= 1 - [P(X_1 < 1500)]^5 \quad \text{since } X_i \sim N(1050, 150) \text{ for all } i \in \{1, 2, \cdots, 5\}$$

$$= 1 - [P(Z < 3)]^5$$

$$= \boxed{0.0065}$$

38. Let $X_1 \sim N(3, 0.1)$ be machine 1's corks.
$X_2 \sim N(3.04, 0.02)$ be machine 2's corks.

$P(2.9 \leq X_1 \leq 3.1) = P(-1 \leq Z \leq 1) = 0.8413 - 0.1587 = 0.6826$

$P(2.9 \leq X_2 \leq 3.1) = P(-7 \leq Z \leq 3) = 0.9987 - 0 = 0.9987.$

$\therefore$ Machine 2 is much more likely to produce acceptable corks.