MATH 4E Fall 2012
Handout for Sections 5.2, 5.3

1. Definition: A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

   where p and q are polynomials and $q(x) \neq 0$. The domain for $R(x)$ is the set of all real numbers except those where q is 0.

2. Definition: Let R be a function.
   If as $x \to -\infty$ or $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
   If as x approaches some number c the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R.

3. Facts:

   - A horizontal asymptote describes the end behavior of the graph as $x \to \infty$ or $x \to -\infty$. The graph of a function may intersect a horizontal asymptote.
   - A vertical asymptote describes the behavior of the graph when x is close to c. The graph of a rational function will never cross a vertical asymptote.
   - If as $x \to \infty$ or $x \to -\infty$ the value of $R(x)$ approaches a linear expression $ax + b, a \neq 0$, then the line $y = ax + b$ is an oblique asymptote of R. The graph of R may intersect an oblique asymptote.

4. Theorem: **Locating Vertical Asymptotes**
   A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, will have vertical asymptote $x = r$ if r is a real zero of $q(x)$, ie if $x - r$ is a factor of $q(x)$.

5. Theorem: If a rational function is proper—the degree of the numerator is less than the degree of the denominator—then $y = 0$ is a horizontal asymptote of its graph.

6. **Finding Horizontal or Oblique Asymptotes**:
   Consider

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

   in which the degree of the degree of the numerator is n and the degree of the denominator is m.

   - If $n < m$ then R is a proper rational function and the graph of R will have a vertical asymptote at $y = 0$ (the x-axis).
   - If $n \geq m$ then R is improper and we must use long division
     - If $n = m$ then the quotient obtained will be $\dfrac{a_n}{b_m}$ and the line $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote.
     - If $n = m + 1$ the quotient obtained is of the form $ax + b$ (a linear equation) and the line $y = ax + b$ is an oblique asymptote.
     - If $n \geq m + 2$ the quotient obtained is a polynomial of degree two or higher and R has neither a horizontal nor oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

   Note: The graph of a rational function either has one horizontal or one oblique asymptote or has no horizontal and no oblique asymptote.

7. Analyzing the Graph of a Rational Function R:

(a) Factor the numerator and denominator in R. Find the domain.

(b) Write R in lowest terms.

(c) Locate intercepts. The x-intercepts must be in the domain. Determine the behavior of R at each intercept.

(d) Determine the vertical asymptotes. Graph each with a dashed line.

(e) Determine horizontal or oblique asymptotes, if one exists. Determine points, if any, where R intersects the asymptote. Graph the asymptote with a dashed line, marking points where R intersects it.

(f) Use the zeros of the numerator and denominator to divide the x-axis into intervals. Pick an x in each interval and determine where R is above and below the x-axis by evaluating R at your chosen x. Plot these.

(g) Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.