# Math 1160A — Exam 2

Name: KEY

Friday, July 8, 2016
Time: 60 minutes
Instructor: Brittany Cuchta

**Instructions:**

- Do not open the exam until I say you may.
- All cell phones and other electronic noisemaking devices must be turned off or completely silenced (i.e., not on vibrate) for the duration of the exam.
- **No calculators** are allowed on the exam.
- The exam *must* be taken in pencil. Using a pen on the exam will result in the loss of points.
- Failure to follow directions specific to a problem will result in the loss of points.
- Circle or box your final answer where appropriate. Put your final answer in the provided space when available. Failure to do so will result in points being deducted.
- Show **all** work. Full credit will only be given if work is shown which **fully and clearly** justifies your answer. I reserve the right to not grade a problem which I cannot read.
- Answers must be exact (like $\sqrt{2}$), not approximate (like 1.414), unless a problem specifically indicates otherwise.
- All final answers must be simplified unless otherwise specified. **Rationalization is not required unless otherwise specified.**
- If you run out of room, use the back of the page and indicate this on the question.
- As always, you are expected to exhibit academic integrity during the exam.

| Page: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 24 | 32 | 19 | 22 | 97 |
| Score: | | | | | |

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\sin\alpha + \sin\beta = 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\tfrac{\alpha-\beta}{2}\cos\tfrac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

1. (10 points) Find the inverse function $f^{-1}$. State the domain and range of $f^{-1}$ using interval notation.

$$f(x) = -2\sin\left(x + \frac{\pi}{3}\right) + 2, \qquad -\frac{5\pi}{6} \le x \le \frac{\pi}{6}$$

$$y = -2\sin(x + \tfrac{\pi}{3}) + 2$$

$$x = \sin^{-1}\left(-\frac{y-2}{2}\right) - \frac{\pi}{3}$$

$f^{-1}(y) =$ $\sin^{-1}\left(\frac{2-y}{2}\right) - \frac{\pi}{3}$

$D_{f^{-1}}$ : $[0, 4]$

$R_{f^{-1}}$ : $\left[-\frac{5\pi}{6}, \frac{\pi}{6}\right]$

2. (8 points) Prove the following identity:

$$\sec(2\theta) = \frac{\sec^2\theta}{2 - \sec^2\theta}$$

$$\text{RHS} = \frac{\sec^2\theta}{2-\sec^2\theta} = \frac{\frac{1}{\cos^2\theta}}{2 - \frac{1}{\cos^2\theta}} = \frac{\frac{1}{\cos^2\theta}}{\frac{2\cos^2\theta - 1}{\cos^2\theta}}$$

$$= \frac{1}{2\cos^2\theta - 1} = \frac{1}{\cos(2\theta)} = \sec(2\theta) = \text{LHS}.$$

3. (6 points) Circle true or false for each statement. No partial credit will be given.

(a) True False : Since $\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{5\pi}{4}$.

(b) True False : The domain of the inverse sine function is all real numbers.

(c) True False : The range of the inverse tangent function is $[-1, 1]$.

4. Find the exact value of the following expressions.

(a) (4 points) $\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right]$

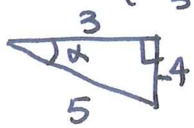
$= \sin^{-1}\left(\sin\left(\frac{2\pi}{5}\right)\right)$

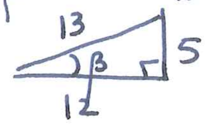
Solution: $\frac{2\pi}{5}$

(b) (10 points) $\cos\left[\tan^{-1}\left(-\frac{4}{3}\right) - \sin^{-1}\left(\frac{5}{13}\right)\right]$ $\alpha$ $\beta$

$\alpha = \tan^{-1}\left(-\frac{4}{3}\right)$ $\beta = \sin^{-1}\left(\frac{5}{13}\right)$

$\alpha \in QIV$




$\beta \in QI$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{3}{5} \cdot \frac{12}{13} + \frac{-4}{5} \cdot \frac{5}{13}$

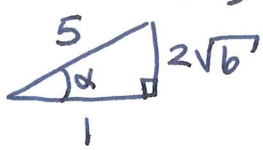
$= \frac{36 - 20}{65}$

$= \frac{16}{65}$

Solution: $\frac{16}{65}$

(c) (8 points) $\csc\left[2 \cdot \cos^{-1}\left(\frac{1}{5}\right)\right]$ $\alpha$

$\alpha = \cos^{-1}\left(\frac{1}{5}\right)$




$\alpha \in QI$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(\frac{2\sqrt{6}}{5}\right)\left(\frac{1}{5}\right) = \frac{4\sqrt{6}}{25}$

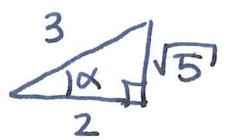
$\therefore \csc(2\alpha) = \frac{25}{4\sqrt{6}}$

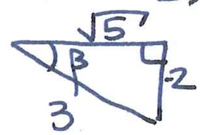
Solution: $\frac{25}{4\sqrt{6}}$

(d) (10 points) $\sin\left[\cos^{-1}\left(\frac{2}{3}\right) + \csc^{-1}\left(-\frac{3}{2}\right)\right]$ $\alpha$ $\beta$

$\alpha = \cos^{-1}\left(\frac{2}{3}\right)$, $\beta = \csc^{-1}\left(-\frac{3}{2}\right)$

$\alpha \in QI$




$\beta \in QIV$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \frac{\sqrt{5}}{3} \cdot \frac{\sqrt{5}}{3} + \frac{-2}{3} \cdot \frac{2}{3}$

$= \frac{5 - 4}{9} = \frac{1}{9}$

Solution: $\frac{1}{9}$

5. Solve the following equations on the interval $[0, 2\pi)$.

(a) (12 points) $\cos(6x) + \sin^2(3x) = 0$

$$\cos(2 \cdot 3x) + \sin^2(3x) = 0$$

$$1 - 2\sin^2(3x) + \sin^2(3x) = 0$$

$$1 - \sin^2(3x) = 0$$

$$\sin^2(3x) = 1$$

$$\sin(3x) = \pm 1$$

$$\begin{cases} 3x = \frac{\pi}{2} + 2k\pi \\ 3x = \frac{3\pi}{2} + 2k\pi \end{cases} \rightarrow \begin{cases} x = \frac{\pi}{6} + \frac{2}{3}k\pi \\ x = \frac{\pi}{2} + \frac{2}{3}k\pi \end{cases}$$

$\underline{k=0}$: $x = \frac{\pi}{6}$

$x = \frac{\pi}{2}$

$\underline{k=1}$: $x = \frac{\pi}{6} + \frac{4\pi}{6} = \frac{5\pi}{6}$

$x = \frac{3\pi}{6} + \frac{4\pi}{6} = \frac{7\pi}{6}$

$\underline{k=2}$: $x = \frac{\pi}{6} + \frac{8\pi}{6} = \frac{9\pi}{6}$

$x = \frac{3\pi}{6} + \frac{8\pi}{6} = \frac{11\pi}{6}$

Solution: $\left\{ \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$

(b) (7 points) $2\sin(\theta) + \sin(2\theta) = 0$

$$2\sin\theta + 2\sin\theta\cos\theta = 0$$

$$2\sin\theta(1 + \cos\theta) = 0$$

$\sin\theta = 0$ or $1 + \cos\theta = 0$

$\theta = 0 + 2k\pi$ $\qquad$ $\theta = \pi + 2k\pi$

$\theta = \pi + 2k\pi$

Solution: $\{0, \pi\}$

6. (8 points) Give the general solution to the following:

$$\tan(2\theta) = 1$$

$$\begin{cases} 2\theta = \frac{\pi}{4} + 2k\pi \\ 2\theta = \frac{5\pi}{4} + 2k\pi \end{cases} \rightarrow \begin{cases} \theta = \frac{\pi}{8} + k\pi \\ \theta = \frac{5\pi}{8} + k\pi \end{cases}$$

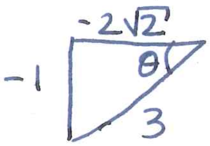
or: $2\theta = \frac{\pi}{4} + k\pi \rightarrow \theta = \frac{\pi}{8} + \frac{\pi}{2}k$

Solution: $\left\{ \frac{\pi}{8} + k\pi, \frac{5\pi}{8} + k\pi \right\}$

or $\left\{ \frac{\pi}{8} + \frac{\pi}{2}k \right\}$

7. Find the exact value of the following.

(a) (8 points) If $\csc\theta = -3$ and $\cos\theta < 0$, find $\cos\left(\frac{\theta}{2}\right)$.




$\csc\theta = -3 \rightarrow \sin\theta = -\frac{1}{3}$, $\theta \in Q\,III$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{-2\sqrt{2}}{3}}{2}}$$

$$= -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

$\pi \le \theta \le \frac{3\pi}{2}$

$\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$

$\rightarrow \frac{\theta}{2} \in Q\,II$

Solution: $-\sqrt{\frac{3-2\sqrt{2}}{6}}$

(b) (6 points) $\sin\left(-\frac{\pi}{12}\right)$

$$\sin\left(-\frac{\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right) = -\sin\left(\frac{1}{2}\cdot\frac{\pi}{6}\right) = -\sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$$

or: $-\sin\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = -\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= -\sin\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos\frac{\pi}{4}$

$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

Solution: $-\frac{\sqrt{2-\sqrt{3}}}{2}$ or $\frac{\sqrt{2}-\sqrt{6}}{4}$