§8.7 – Product-to-Sum and Sum-to-Product Formulas

Sum and difference formulas can be used to derive ways to write product of trig equations as sum or vice versa.

Product-to-Sum:

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

Sum-to-Product:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Examples: Express the following products as a sum containing only sines or cosines.

A. $\cos(2\theta)\cos(4\theta)$

Soln: "α"$= 2\theta$, "β"$= 4\theta$ so

$$\begin{aligned}\cos(2\theta)\cos(4\theta) &= \frac{1}{2}\left[\cos(2\theta-4\theta) + \cos(2\theta+4\theta)\right]\\ &= \frac{1}{2}\left[\cos(-2\theta) + \cos(6\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta) + \cos(6\theta)\right]\end{aligned}$$

B. $\sin(3\theta)\sin(5\theta)$

Soln: "α"$= 3\theta$, "β"$= 5\theta$ so

$$\begin{aligned}\sin(3\theta)\sin(5\theta) &= \frac{1}{2}\left[\cos(3\theta - 5\theta) - \cos(3\theta + 5\theta)\right]\\ &= \frac{1}{2}\left[\cos(-2\theta) - \cos(9\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta) - \cos(8\theta)\right]\end{aligned}$$

Examples: Express each sum or difference as a product of sines and/or cosines.

C. $\sin(5\theta) - \cos(3\theta)$

Soln: "α"$= 5\theta$, "β"$= 3\theta$ so

$$\begin{aligned}\cos(5\theta) - \cos(3\theta) &= -2\sin\left(\frac{5\theta + 3\theta}{2}\right)\sin\left(\frac{5\theta - 3\theta}{2}\right)\\ &= -2\sin\left(\frac{8\theta}{2}\right)\sin\left(\frac{2\theta}{2}\right)\\ &= -2\sin(4\theta)\sin(\theta)\end{aligned}$$

D. $\sin(2\theta) + \sin(4\theta)$

Soln: "α"$= 2\theta$, "β"$= 4\theta$ so

$$\begin{aligned}\sin(2\theta) + \sin(4\theta) &= 2\sin\left(\frac{2\theta + 4\theta}{2}\right)\cos\left(\frac{2\theta - 4\theta}{2}\right)\\ &= 2\sin\left(\frac{6\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)\\ &= 2\sin(3\theta)\cos(-\theta)\\ &= 2\sin(3\theta)\cos\theta\end{aligned}$$

You will **not** need to memorize these formulas for the exam: they will be given to you. You will only need to know when to use which formula.

Use the following exercises as practice.

Exercise 1: $\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$

Exercise 2: $\cos\frac{5\theta}{2}\sin\frac{\theta}{2}$

Exercise 3: $\cos(2\theta) + \cos(4\theta)$

Exercise 4: $\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}$

Exercise 5: Prove that $\frac{\sin(4\theta)+\sin(8\theta)}{\cos(4\theta)+\cos(8\theta)} = \tan(6\theta)$