Zero Factorial

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So, I had a couple people ask about this, and it's a good question! *Why is* $0! = 1$ *?* There are a couple ways to think about this.

The first way is to notice a pattern. Since "factorial" is traditionally defined as multiplying numbers down to one, zero would certainly pose a problem since it's *already* less than one. So we'll make a new formula! Consider 4!. We will use the "convenient form of the number one" again here, as we did in class when deriving the permutation formula.

$$\begin{aligned} 4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot \tfrac{5}{5} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5} \\ &= \frac{5!}{5} \end{aligned}$$

Clearly, we could do this same idea for any given number n so we may say that in general,

$$n! = \frac{(n+1)!}{n+1}$$

Try this out for a few values if you're unconvinced.

With our new formula in hand, we can determine the value of 0!.

$$\begin{aligned} 3! &= \frac{4!}{4} = \frac{24}{4} = 6 \\ 2! &= \frac{3!}{3} = \frac{5}{3} = 2 \\ 1! &= \frac{2!}{2} = \frac{2}{2} = 1 \\ 0! &= \frac{1!}{1} = \frac{1}{1} = 1 \end{aligned}$$

and thus we obtain the desired result.[1]

If this method felt shady to you (I understand if it did, though it is a valid derivation for 0!), then consider placing kittens in baskets.

[1]Note that using this method we can see that $(-1)! = \frac{0!}{0}$ is undefined. There is in fact an extension of the factorial function, called the Gamma function and denoted $\Gamma(n) = (n-1)!$. And guess what? $\Gamma(0)$ is undefined! More specifically, the limit from the left approaches $-\infty$ while the limit from the right approaches $+\infty$. If you're interested in more information about the lovely gamma function, let me know and I will provide resources.

We have four different kittens we need to put in four baskets, numbered 1 through 4. The first basket has four choices, the second basket has three, the third basket has two, and the fourth basket has one choice, giving us a total of 24 ways to put the kittens in the baskets. This is exactly our "product rule" for k-tuples! So, if 4! represents putting four kittens in baskets, and in general n! represents putting n kittens in baskets, what would 0! be?: How many ways to put *no kittens* in baskets. And how many ways could we do this? Just one! There's only one way to *not* do something, and that's to not do it!

I hope this exposé was enlightening for you and hopefully makes you feel better about 0!. So, while it's useful for 0! to be one instead of zero for calculations' sake, it's also for an actual *reason*, which I hope I've satisfactorily convinced you of!